CRITICAL VALUES OF FIXED MORSE INDEX OF RANDOM ANALYTIC FUNCTIONS ON RIEMANN SURFACES

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This note is an addendum to [FZ]. In that article, we determined the limit distribution of *critical values* of pointwise norms $|s_n(z)|_{h^n}$ of L^2-normalized random global holomorphic sections of $H^0(M, L^n)$ where (M, ω) is a Kähler manifold of complex dimension m and $(L, h) \to (M, \omega)$ is a positive holomorphic line bundle of degree 1 whose curvature form is $\omega = \text{Ric(h)}$. The line bundle L^n is the nth power of L, so that $H^0(M, L^n)$ is analogous to the space of polynomials of degree n and the Hermitian metric h^n on L^n is the nth tensor power of h. We studied two probability measures on $H^0(M, L^n)$: (i) a certain canonical "normalized Gaussian measure" induced by h, for which $\mathbf{E}\,||s||_{L^2} = 1$; and (ii) the Euclidean surface 'spherical measure' on the unit sphere $SH^0(M, L^n)$ where $||s||_{L^2} = 1$. The motivation for use of the spherical measure was to count a critical value just once in a family of dilates $\{cs : c \in \mathbb{C}\}$. In this note, we add two results on the large n asymptotics of the density of critical values of norms of sections in $H^0(M, L^n)$ to [FZ] which clarify the nature of the results.

The first addition is to make more precise Remark 2 of [FZ], where it was explained how to compute the distribution of the critical values at critical points of $|s_n(z)|_{h^n}$ of fixed Morse index. The calculation in general complex dimension m is a complicated Kac-Rice integral over complex symmetric matrices of fixed index and rank m in complex dimension m. But in complex dimension one (i.e. on Riemann surfaces), it is simple enough to evaluate explicitly. The first purpose of this note is to supply the calculation of the limit distribution of critical values at local maxima or at saddles on Riemann surfaces, completing Remark 2 of [FZ].

The second addition is the calculation of the second order term in the large n asymptotic expansion of the expected density of critical values on Riemann surfaces. The result is that the second term is a topological term. It follows that the large n expansion is universal to two orders. In [DSZ] the same kind of universality was shown for the density of critical points on M.

Throughout we assume familiarity with the notation and terminology of [FZ]. To keep the article to an appropriate length, we only review notation and results that are needed to state and prove the new results.

0.1. **Density of critical values at local maxima and saddles.** In [DSZ, DSZ1] the authors determined the distribution of critical points of a fixed Morse index, and the present discussion takes off from that point. Note that $d|s_n|^2_{h_n} = 0 \Leftrightarrow \nabla_{h^n} s_n = 0$ or $s_n = 0$. Local minima of $|s_h|^2$ are necessarily zeros, and thus are trivial from the viewpoint of critical values. As in [DSZ1], the topological index of a section s at a critical point z_0 is defined to be the index of the vector field $\nabla_h s$ at z_0 (where $\nabla_h s$ vanishes). Critical points of a section s in dimension one are (almost

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surely) of topological index ± 1. The critical points of s of index 1 are the saddle points of $\log |s|_h$ (or equivalently, of $|s|_h^2$), while those of topological index -1 are local maxima of $\log |s|_h$ in the case where L is positive. In complex dimension one, and with (L, h) a positive line bundle, topological index 1 corresponds to $\log |s|_h$ having Morse index 1, while topological index -1 corresponds to Morse index 2. In [DSZ1] it is shown that in complex dimension one, the number of saddle points of $|s|_{h^n}^2$ is asymptotically $\frac{4}{3}n$, while the number of local maxima is asympotically $\frac{1}{3}n$.

Let $\omega = \frac{i}{2}\partial\bar{\partial} \log h$ be the Kähler metric associated to (L, h). Thus, $\frac{1}{\pi}\omega$ is a de Rham representative of the Chern class $c_1(L) \in H^2(M, \mathbb{R})$[1].

In any dimension m, we define the empirical measure of nonvanishing critical values of index k as

$$CV_{s_n}^{m,k} = \frac{1}{n^m} \sum_{z:\, \nabla_{h^n} s=0 \text{ with index } k} \delta_{|s_n|_{h^n}}$$

and define the expected distribution of such critical values by

$$\mathbf{D}_n^{m,k}(x) := \mathbf{E}\,(CV_{s_n}^{m,k}).$$

As seen below, it is a measure with a smooth density on $\mathbb{R}$. Then as proved in [FZ] (see Remark 2),

Theorem 1. [FZ] *For both the normalized Gaussian ensemble and spherical ensemble, the universal limit as $n \to \infty$ for the expected density of nonvanishing critical values of index k,*

$$\lim_{n\to\infty} \mathbf{D}_n^{m,k}(x) = f_{m,k}(x) x e^{-\pi^m x^2},$$

where

$$f_{m,k}(x) = c_m \int_{S_{k,x}(\mathbb{C}^m)} e^{-\pi^m |\xi|^2} \left| |\sqrt{P}\xi|^2 - x^2 I \right| d\xi$$

where

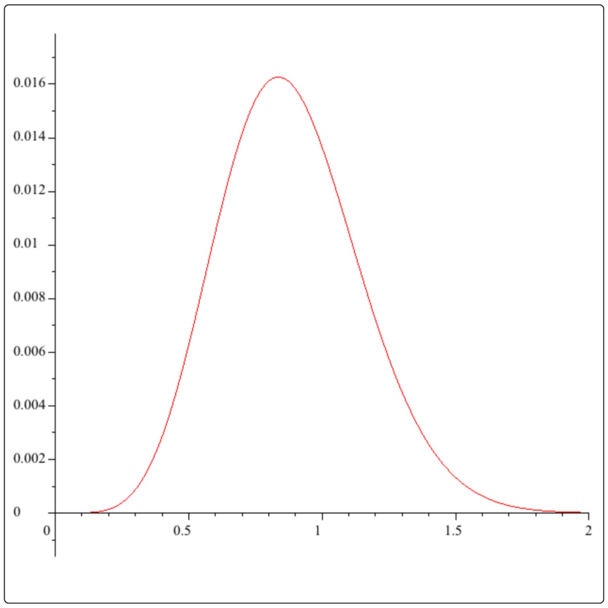
$$S_{k,x}(\mathbb{C}^m) = \{\xi \in S(\mathbb{C}^m) :\ \mathrm{index}(|\sqrt{\mathrm{P}}\xi|^2 - \mathrm{x}^2\mathrm{I}) = \mathrm{k}\}.$$

We now evaluate the integrals in complex dimension 1.

In the case of Riemann surfaces, the matrix $P = 2$. Assuming the degree of L equals 1, the area $\int_M \omega = \pi$. Then the limit of the expected density of local maxima of $|s|_{h^n}$ for $s \in H^0(M, L^n)$ is given by the universal formula,

$$\begin{aligned}\lim_{n\to\infty} \mathbf{D}_n^{1,-1}(x) &= x \left(\frac{2}{\pi} \int_{\xi\in\mathbb{C}:\, |\xi|^2 < \frac{x^2}{2}} e^{-\pi|\xi|^2} (x^2 - 2|\xi|^2) d\xi \right) e^{-\pi x^2} \\ &= x \left(\frac{2}{\pi} x^2 - \frac{4}{\pi^2} + \frac{4}{\pi^2} e^{-\frac{\pi x^2}{2}} \right) e^{-\pi x^2}.\end{aligned}$$

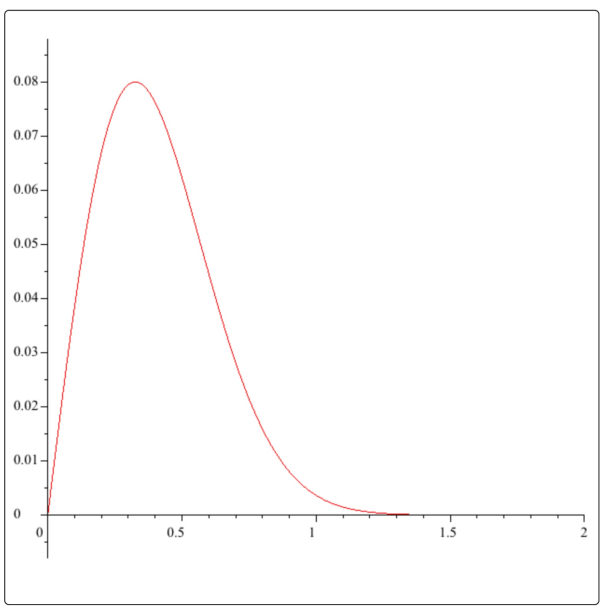
[1] In [FZ], the area form was defined to be ω/π. In this article we define it to be ω. As a result, there are slight differences in normalization between this article and [FZ].



Graph of $\mathbf{D}_\infty^{1,-1}(x)$ in dimension one

Moreover, the expected density of saddle values has the universal limit,

$$\lim_{n\to\infty} \mathbf{D}_n^{1,1}(x) = x\left(\frac{2}{\pi}\int_{\xi\in\mathbb{C}:\,|\xi|^2>\frac{x^2}{2}} e^{-\pi|\xi|^2}(2|\xi|^2 - x^2)d\xi\right)e^{-\pi x^2}$$
$$= \frac{4x}{\pi^2}e^{-\frac{3\pi}{2}x^2}.$$



Graph of $\mathbf{D}_\infty^{1,1}(x)$ in dimension one

In both cases, the density of critical values x at saddles and local maxima vanishes to order one at $x = 0$. Although we take the norm (or modulus) when defining critical values, the density reflects the fact that the sections are complex valued and x plays the role of the polar coordinate.

0.2. **The second order topologically invariant term.** Theorems 1.1 and 1.5 of [FZ] give the leading term in the density of critical values. In fact, the proof shows that there exists a full expansion

$$\mathbf{D}_n = \mathbf{D}_\infty + \frac{1}{n}\mathbf{F}_\infty + \cdots$$

in powers of n. The leading order term $\mathbf{D}_\infty$ is calculated in [FZ] and shown to be universal. The second addition of this note is to prove the topological invariance of the second term in complex dimension one:

Theorem 2. *For both the normalized Gaussian and spherical ensembles, the second order term is a topological invariant of Kähler metrics on Riemann surfaces,*

$$\mathbf{F}_\infty(x) = -\frac{\chi(M)\pi^2 x}{4}[\int_{\mathbb{C}} e^{-\frac{\pi}{2}|\xi|^2-\pi x^2}(\pi|\xi|^2-2)\left||\xi|^2-x^2\right| d\xi]$$

where $\chi(M)$ is the Euler characteristic of the Riemann surfaces.

Proof. The Kac-Rice formula for $\mathbf{D}_n$ is given in Section 4 (page 665 of [FZ]) in terms of a certain function $p^n_z(x,\theta,0,\xi)$ given in (5.2) and Lemma 5.1. To determine the full expansion it is only necessary to find the expansion of the matrices A_n and Λ_n (see Section 7.2). The entries of the matrices are various derivatives of the Bergman kernel $B_n(z,w)$ on the diagonal. We recall the TYZ expansion (cf. [Lu]),

$$B_n(z,z) = \frac{1}{\pi^m} n^m e^{n\varphi(z)}\left[1 + a_1(z)n^{-1} + a_2(z)n^{-2} + \cdots\right], \tag{1}$$

where $a_1 = \frac{1}{2}S$ is half of the complex scalar curvature S of ω, and is a quarter of the scalar curvature of the Riemannian metric[2]. Also, $h = e^{-\varphi}$ in a local frame. Denote by d_n the dimension of $H^0(M, L^n)$. By Riemann-Roch, in complex dimension one, $d_n = n + 1 - g = n + \frac{1}{2}\chi(M)$, where $\chi(M) = 2 - 2g$ is the Euler characteristic. This also follows by integrating the $e^{-n\varphi}$ times (1) (i.e. the Szegö kernel $\Pi_n(z,z)$) and applying Gauss- Bonnet.

In Section 7 of [FZ], we used the estimates of covariance matrix to get the expression of the leading term $\mathbf{D}_\infty$. Here, we continue with the estimates to get the expression for $\mathbf{F}_\infty$ on Riemann surfaces. We only keep track of terms that contribute to the second order term and ignore all negligible terms. Recall in Sections 4 and 5 of [FZ], we have the following formula for $\mathbf{D}_n$ in the case of Riemann surfaces with $m = 1$,

$$\mathbf{D}_n(x) = \frac{2x}{\pi^2 n}\int_M \int_{\mathbb{C}} \frac{e^{-\left\langle \begin{pmatrix}\xi\\ y\end{pmatrix}, \Lambda_n^{-1}\begin{pmatrix}\bar\xi\\ \bar y\end{pmatrix}\right\rangle}}{A_n \det \Lambda_n}\left||\xi|^2 - n^2x^2\right| d\xi dV_\omega,$$

where

$$A_n = \frac{n}{\pi d_n}\left(nI + a_1 I + n^{-1}(a_2 I + \partial_j\bar\partial_{j'}a_1 + \cdots)\right)$$

[2] We thank Z. Lu for clarifying the coefficient of a_1 in (1).

and by (1),

$$\Lambda_n = \frac{n}{\pi d_n}\begin{pmatrix} (2n^2 - nS)(1 + n^{-1}a_1 + \cdots) & n^{-1}\partial_j\partial_q a_1 + O(n^{-2}) \\ n^{-1}\bar{\partial}_j\bar{\partial}_q a_1 + O(n^{-2}) & 1 + n^{-1}a_1 + O(n^{-2}) \end{pmatrix}.$$

We change variable $\xi \to n\xi$ to rewrite,

$$\mathbf{D}_n(x) = \frac{2n^3 x}{\pi^2}\int_M\int_{\mathbb{C}} \frac{e^{-\left\langle \begin{pmatrix}\xi\\ y\end{pmatrix}, \frac{\pi d_n}{n}\tilde{\Lambda}_n^{-1}\begin{pmatrix}\bar{\xi}\\ \bar{y}\end{pmatrix}\right\rangle}}{A_n \det \Lambda_n}\left||\xi|^2 - x^2\right| d\xi dV_\omega. \tag{2}$$

Here, $\tilde{\Lambda}_n$ has the full expansion

$$\tilde{\Lambda}_n = \Lambda^0 + n^{-1}\Lambda^1 + \cdots$$

with

$$\Lambda^0 = \begin{pmatrix} 2 & 0 \\ 0 & 1\end{pmatrix}, \quad \Lambda^1 = \begin{pmatrix} 2a_1 - S & 0 \\ 0 & a_1\end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & a_1\end{pmatrix}$$

where we use the fact that the second term a_1 in TYZ expansion equals to $\frac{1}{2}S$.

Thus

$$\tilde{\Lambda}_n^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1\end{pmatrix} - n^{-1}\begin{pmatrix} 0 & 0 \\ 0 & a_1\end{pmatrix} + \cdots$$

It follows that,

$$e^{-\left\langle \begin{pmatrix}\xi\\ y\end{pmatrix}, \frac{\pi d_n}{n}\Lambda_n^{-1}\begin{pmatrix}\bar{\xi}\\ \bar{y}\end{pmatrix}\right\rangle}$$

$$= e^{-\frac{\pi}{2}|\xi|^2 - \pi x^2}\left(1 - \pi n^{-1}[\frac{1}{4}\chi(M)|\xi|^2 + (\frac{1}{2}\chi(M) - a_1)x^2] + \cdots\right).$$

If we substitue the asymptotic expansions of A_n and Λ_n into the equation (2), and only keep track of terms of order n^{-1}, we get

$$\mathbf{F}_\infty = -\pi^2 x\int_M\int_{\mathbb{C}} e^{-\frac{\pi}{2}|\xi|^2 - \pi x^2}\left([\frac{1}{4}\chi(M)|\xi|^2 + (\frac{1}{2}\chi(M) - a_1)x^2]\right)$$

$$||\xi|^2 - x^2| d\xi dV_\omega + \pi x\int_M\int_{\mathbb{C}}(\frac{3}{2}\chi(M) - 3a_1 + \frac{1}{2}S)e^{-\frac{\pi}{2}|\xi|^2 - \pi x^2}||\xi|^2 - x^2| d\xi dV_\omega.$$

Recall Gauss-Bonnet Theorem $\frac{\pi}{2}\chi(M) = \int_M a_1\omega$ again, combine this with the assumption that the volume of M is π, we have $\int_M(\frac{1}{2}\chi(M) - a_1)\omega = 0$. Thus we can simplify the above expression as,

$$\mathbf{F}_\infty(x) = -\frac{\chi(M)\pi^3 x}{4}\int_{\mathbb{C}} e^{-\frac{\pi}{2}|\xi|^2 - \pi x^2}|\xi|^2\left||\xi|^2 - x^2\right| d\xi$$

$$+\frac{\chi(M)\pi^2 x}{2}\int_{\mathbb{C}} e^{-\frac{\pi}{2}|\xi|^2 - \pi x^2}\left||\xi|^2 - x^2\right| d\xi$$

which completes the proof.

$\square$

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